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Theory of propagation of cracks

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SUMMARY

The criterion for the plastic flow of crystalline solids is well established but a similar criterion for the fracture of brittle solids under triaxial stress conditions has not been proposed so far. According to the present theory, brittle fracture occurs as the result of the formation and subsequent propagation of microcracks. In this paper it is shown that the thermodynamic criterion for crack propagation is not a sufficient one and leads to unsatisfactory results in the general case. The necessary and sufficient criterion that there must be a local stress sufficient to rupture the atomic bonds at the edge of the crack does lead to satisfactory results. Griffith's crack is taken as a model and a calculation is carried out for the following boundary conditions: (1) at large distance from the crack there is an arbitrary plane stress or strain field; (2) at the crack boundary the crack surface is free from traction. This theory leads to a parabolic relationship

$$\tau_{12}^2 + 4K'\tau_{22} = 4K'^2$$

between the shear stress τ_{12} and the normal stress τ_{22} acting on the plane containing the propagating Griffith crack.

1. Introduction

The theory of brittle fracture has been developed by Griffith [1] and subsequently by others in terms of the propagation and formation of microcracks in solids. In most of these works, the authors considered fractures under the condition of uniaxial tension only. But it is well known that some important brittle solids develop shear fractures, often when the three principal stresses are compressive. One should therefore consider the problem of formation and propagation of microcracks in bodies under the most general conditions. A microcrack will propagate when the stresses are sufficient to rupture the atomic bonds at the edge of the crack. The calculation of the strength of the atomic bonds is not a straightforward one and an alternative thermodynamic approach was initiated by Griffith [1]. According to thermodynamics the condition for crack propagation can be deduced from the condition that there should be a reduction of the Gibb's free energy G of the system consisting of the body containing the crack together with the straining agent. However, on physical grounds it is obvious that the thermodynamic condition may not be a sufficient one. In this paper we imagine the body to be filled with identical small cracks which pierce the plane of the body and which have a cross section in the form of an ellipse parallel to this plane, the ratio of the minor to the major axis of the ellipse being very small. The cracks are supposed to be spaced sufficiently far apart, so that the presence of other cracks has only a minor effect on the stress field around any particular crack. Tractions are applied to the body so as to produce an arbitrary but uniform condition of plane strain or plane stress at points sufficiently far from the cracks. The orientation of the ellipse axes of the cracks are supposed to have a random distribution with respect to the axes of the uniform principal stresses produced by the applied tractions. The conditions are calculated at which rupture will occur at the surface of similarly oriented cracks. This latter process has been described as the propagation of cracks.

If the parameters describing the cracks, namely the length of the crack and the radius of curvature of its apex are sufficiently large compared with the equilibrium interatomic distance b it is clear that the stresses and displacements calculated by the mathematical theory of elasticity will not be strongly dependent upon the arrangement of atoms around the crack. Also, Inglish [2] and Neuber [3] have pointed out that the stress concentration produced by a long narrow crack depends on the length of the crack and the radius of curvature of its apices but it is very little dependent on the precise shape of the crack at points some distance away from the apices. The results given in this paper are therefore certainly valid.

As the radius of curvature of the apex of the crack takes smaller and smaller values a stage is reached at which the arrangement of atoms in and around the crack surface and especially near the apex of the crack cannot be ignored. Stress and strain do not vary continuously. Within the small volume of approximately b^3 between pairs of atoms, the stress and strain may be regarded as constant and the changes of stress and strain between neighbouring small volumes are discontinuous. If the change of stress and strain over distances of the order of b measured from a given point are a small portion of the mean value in the neighbourhood of the point then the error in the stress and strain at the point calculated from continuum theory of elasticity will be small. In the case of sharp cracks in a real solid the maximum tensile stress must not exceed the maximum tensile interatomic force per unit area which is denoted by T.

As the radius of curvature of the apex tends to zero, an elliptic crack becomes a slit crack (i.e. a part of a plane across which tensile stresses are not transmitted). The limiting case among real cracks is that of a cleavage crack, formed by the splitting apart of planes of atoms which were formerly bonded together. But when tensile tractions are applied to the body, infinite tensile stresses appear at the apices of the crack. This makes the problem difficult.

Elliot [4] considered the case of a slit crack separating two parallel planes of atoms in a large body subject to a tensile stress acting normal to the crack, which is uniform at a large distance from the crack. He established the condition for rupture (i.e. propagation of the crack) as that the maximum tensile force per unit area acting on the planes of atoms closest to the surface of a crack must be equal to the maximum tensile interatomic force per unit area T. He found a value of T by using the fact that the area under the $f(x_2)$ curve is equal to twice the surface energy v where $f(x_2)$ is the force per unit area between atom planes at a distance x_2 apart.

Barenblatt [5] has shown that if one of the boundary conditions is that the stress is constant across a region of the plane of the slit crack, then there is a value for the width of the region which is such that the stress at the edge of the region does not become infinite. He showed that the condition for the stresses to be non-singular is that they correspond to a minimum of the potential energy of the system with respect to variations of the parameters of the system (crack length, width of constant stress region, value of constant stress). His condition, therefore, determines the equilibrium configuration of the crack under given applied tractions and not for the propagation of cracks.

The works of Leonov and Onyshko [9], Neuber [3], Orwan [6], Cottrell [7, 8] suggest that a cleavage crack may be represented either by a slit crack with regions of uniform stress at its tips or by a narrow elliptic crack with a finite radius of curvature ρ at its tips and the value of ρ will be at least $0.81b$.

2. Condition for propagation of a crack

A crack will propagate when the maximum stress at its edge reaches a value sufficient to rupture the atomic bonds there. The surface energy v was defined by Elliott [4] as

$$2v = \int_b^\infty f(x_2)\,\mathrm{d}x_2\,, \tag{2.1}$$

where $f(x_2)$ is the force per unit area between atom planes at a distance x_2 apart. He derived the following expression for the maximum tensile interatomic force per unit area for isotropic solids having Poisson's ratio $\eta = 0.25$,

$$T = \left(\frac{1.09\ vE}{b}\right)^{\frac{1}{2}}. \tag{2.2}$$

A similar equation was given by Orowan [6]. Calculation of a crack propagation condition is therefore reduced to finding the value of the maximum tensile stress at the crack edge and equating this to T.

In the thermodynamical approach, we require to calculate the change in Gibb's free energy G of the system for a virtual propagation of the crack. Three terms contribute to ΔG in an isothermal process of this type. These terms are the elastic energy U stored in body, the potential energy W of the external forces applied to the body and the surface energy νS of the body (where S is the total surface area of the body). During the application to the body of loads which do not exceed the critical load required to propagate the crack there is no change in G since the increase of the elastic energy U is exactly equal to the decrease of the potential energy of the external forces W, and the surface energy does not change. But as soon as the critical load is reached all three energy terms may change. Two cases may be distinguished: (a) the applied load remains constant while the crack propagates, in which case there is an increase of the elastic energy U and νS and the external forces perform work on the body, so that W decreases; (b) the external boundaries of the body remain fixed, so that the external forces do no further work, i.e. W does not change, and in this case there is a decrease of elastic energy U and an increase of νS. We consider the case (a). We consider a plane body with a crack piercing it and denote the surface of the crack by S_1 and the external surface of the body by S_2. We consider two states of the body. In state I, certain surface tractions are applied to S_2 and a set of tractions is applied to S_1, so that the body deforms as though no crack were present. In state II, the same set of tractions are applied to S_2 but no tractions are applied to S_1. The elastic energy will be denoted by U, surface tractions or stresses by τ_{ij}, the corresponding displacements are denoted by u_i, and the projected area of the small surface ds in the jth direction is denoted by ds_j. The tensor notation is used and the suffixes i and j can take values 1, 2 or 3. The superscripts I and II indicate the values of the quantities in states I and II, respectively.

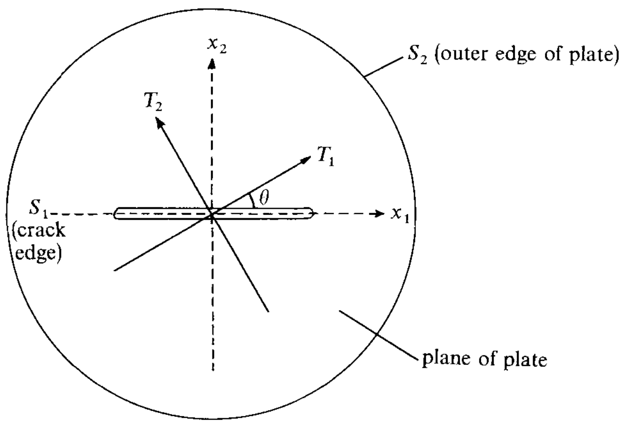

Figure 1. A crack (S_1) in a plane body subjected to principal stresses T_1 and T_2.

The elastic potential energy of a body is given by (Love [10])

$$U = \frac{1}{2}\int_S \tau_{ij} u_i ds_j , \tag{2.3}$$

where the integral is taken over the entire surface of the body and the tractions and displacements are given by their final values. By the Rayleigh–Betti reciprocal theorem we have [18]

$$\frac{1}{2}\int_{S_1+S_2} (\tau_{ij})^{\mathrm{I}}(u_i)^{\mathrm{II}} ds_j = \frac{1}{2}\int_{S_1+S_2} (\tau_{ij})^{\mathrm{II}}(u_i)^{\mathrm{I}} ds_j . \tag{2.4}$$

The boundary conditions

$$((\tau_{ij})^{\mathrm{I}})_{S_2} = ((\tau_{ij})^{\mathrm{II}})_{S_2} , \qquad ((\tau_{ij})^{\mathrm{II}} ds_j)_{S_1} = 0 , \tag{2.5}$$

together with equations (2.3) and (2.4) finally give an expression for the increase of the elastic potential energy of the body produced by the relaxation to zero of the tractions applied to the crack in state I,

$$U(c) = U^{\mathrm{II}} - U^{\mathrm{I}} = -\frac{1}{2}\int_{S_1} (\tau_{ij})^{\mathrm{I}}\left[(u_i)^{\mathrm{I}} + (u_i)^{\mathrm{II}}\right] ds_j . \tag{2.6}$$

This is not quite the same thing as the increase of the elastic potential energy of the body produced by the presence of the traction-free crack, which is

$$U^{\mathrm{II}} - U(0) = -\frac{1}{2}\int_{S_1} (\tau_{ij})^{\mathrm{I}} (u_i)^{\mathrm{II}} ds_j , \tag{2.7}$$

where $U(0)$ is the strain energy of the uncracked body given by

$$U(0) = \frac{1}{2}\int_{S_2} (\tau_{ij})^{\mathrm{I}} (u_i)^{\mathrm{I}} ds_j . \tag{2.8}$$

$U(c)$ and U^{II} are functions of the size and shape of the crack. The parameter determining the size of the crack may be taken as the length $2c$ (of its major diameter) which we shall call the crack length. If there is a virtual increase of crack length of amount $2\Delta c$, with $((\tau_{ij})^{\mathrm{I}})_{S_2}$ and therefore $((\tau_{ij})^{\mathrm{II}})_{S_1}$ remaining constant, then there is an increase of the strain energy of the body which is given by

$$\begin{aligned} \Delta U^{\mathrm{II}} &= -\tfrac{1}{2}\Delta c \frac{\partial}{\partial c}\int_{S_1} (\tau_{ij})^{\mathrm{I}} (u_i)^{\mathrm{II}} ds_j \\ &= -\tfrac{1}{2}\Delta c \int_{S_1} (\tau_{ij})^{\mathrm{I}} \left(\frac{\partial u_i^{\mathrm{II}}}{\partial c}\right) ds_j , \end{aligned} \tag{2.9}$$

since $U(0)$ and τ_{ij} are indepenuent of c. The external forces do an amount of work which is given by

$$-\Delta W = \int_{S_2} (\tau_{ij})^{\mathrm{I}} \left[\left(\frac{\partial u_i^{\mathrm{II}}}{\partial c}\right) \Delta c\right] ds_j . \tag{2.10}$$

From (2.10) it is readily deduced that

$$-\Delta W = -\int_{S_1} (\tau_{ij})^{\mathrm{I}} \left[\frac{\partial u_i^{\mathrm{II}}}{\partial c} \Delta c\right] ds_j = 2\Delta U^{\mathrm{II}} . \tag{2.11}$$

The surface energy of the body increases by an amount

$$\nu \Delta S = \nu \Delta c \frac{\partial s}{\partial c} = \nu \Delta c \frac{\partial s_1}{\partial c} . \tag{2.12}$$

Thus from (2.9), (2.11) and (2.12) we find that the change in the Gibb's free energy of the body for a virtual increase of the crack length is given by

$$\begin{aligned} \Delta G &= \Delta W + \Delta U^{\mathrm{II}} + \nu \Delta S \\ &= -\Delta U^{\mathrm{II}} + \nu \Delta S \\ &= \left[\frac{1}{2}\frac{\partial}{\partial c}\int_{S_1} (\tau_{ij})^{\mathrm{I}} (u_i)^{\mathrm{II}} ds_j + \nu \frac{\partial s_1}{\partial c}\right] \Delta c . \end{aligned} \tag{2.13}$$

For the crack to extend, we have $\Delta G \leqslant 0$, i.e.

$$\frac{1}{2}\frac{\partial}{\partial c}\int_{S_1} (\tau_{ij})^{\mathrm{I}} (u_i)^{\mathrm{II}} ds_j + \nu \frac{\partial s_1}{\partial c} \leqslant 0 . \tag{2.14}$$

3. Condition for the propagation of Griffith's elliptic crack under plane strain

Let us now consider the case of an elliptic crack. We use elliptic co-ordinates (α, β) defined by the transformation formulae

$$\left.\begin{aligned} z = x_1 + ix_2 = c \cosh(\alpha + i\beta) = c \cosh\zeta \\ x_1 = c \cosh\alpha \cos\beta\,, \quad x_2 = c \sinh\alpha \sin\beta \end{aligned}\right\}. \tag{3.1}$$

The major axis is taken as x_1-axis and the x_3-axis is normal to the plane of the body. The curve α = constant is the ellipse

$$\frac{x_1^2}{c^2 \cosh^2\alpha} + \frac{x_2^2}{c^2 \sinh^2\alpha} = 1\,, \tag{3.2}$$

and the elliptic hole has the equation $\alpha = \alpha_0$.

Let T_1 and T_2 be the uniform principal stresses at infinity making angles θ and $\theta + \pi/2$, respectively, with the major axis of the hole. The stresses are then given by (Love [10])

$$\left.\begin{aligned} \tau_{11} &= \tfrac{1}{2}(T_1 + T_2) + \tfrac{1}{2}(T_1 - T_2)\cos 2\theta\,, \\ \tau_{22} &= \tfrac{1}{2}(T_1 + T_2) - \tfrac{1}{2}(T_1 - T_2)\cos 2\theta\,, \\ \tau_{12} &= \tfrac{1}{2}(T_1 - T_2)\sin 2\theta\,. \end{aligned}\right\} \tag{3.3}$$

When there is no hole in the body, Airy's stress function is known to be (Love [10])

$$\Psi = \tau_{22}\tfrac{1}{2}x_1^2 + \tau_{11}\tfrac{1}{2}x_2^2 - \tau_{12}x_1x_2\,. \tag{3.4}$$

The displacements are given by

$$\left.\begin{aligned} u_1 &= \frac{1}{2\mu}\left[-\tau_{22}x_1 + \tau_{12}x_2 + (\tau_{11} + \tau_{22})(1-\eta)x_1\right] \\ u_2 &= \frac{1}{2\mu}\left[-\tau_{11}x_2 + \tau_{12}x_1 + (\tau_{11} + \tau_{22})(1-\eta)x_2\right] \end{aligned}\right\} \tag{3.5}$$

where μ is the shear modulus.

We can now find expressions for the stresses and displacements in elliptic co-ordinates (α, β, x_3). These are (Timoshenko and Goodier [14])

$$\begin{aligned} \tau_{\alpha\alpha}(\alpha_0) &= \tfrac{1}{4}c^2h_0\left[(\tau_{11} - \tau_{22})(\cosh 2\alpha_0 \cos 2\beta - 1) + 2\tau_{12}\sinh 2\alpha_0 \sin 2\beta\right] \\ &\qquad + \tfrac{1}{2}(\tau_{11} + \tau_{22})\,, \\ \tau_{\beta\beta}(\alpha_0) &= -\tfrac{1}{4}c^2h_0\left[(\tau_{11} - \tau_{22})(\cosh 2\alpha_0 \cos 2\beta - 1) + 2\tau_{12}\sinh 2\alpha_0 \sin 2\beta\right] \\ &\qquad + \tfrac{1}{2}(\tau_{11} + \tau_{22})\,, \\ \tau_{\alpha\beta}(\alpha_0) &= \tfrac{1}{4}c^2h_0\left[(\tau_{22} - \tau_{11})\sinh 2\alpha_0 \sin 2\beta + 2\tau_{12}(1 - \cosh 2\alpha_0 \cos 2\beta)\right]\,, \\ u_\alpha(\alpha_0) &= \frac{c^2h_0}{8\mu}\left[(\tau_{11} + \tau_{22})\sinh 2\alpha_0(1 - 2\eta) + (\tau_{11} - \tau_{22})\sinh 2\alpha_0 \cos 2\beta\right. \\ &\qquad \left. + 2\tau_{12}\cosh 2\alpha_0 \sin 2\beta\right]\,, \\ u_\beta(\alpha_0) &= \frac{c^2h_0}{8\mu}\left[-(1 - 2\eta)(\tau_{11} + \tau_{22})\sin 2\beta - (\tau_{11} - \tau_{22})\cosh 2\alpha_0 \sin 2\beta\right. \\ &\qquad \left. + 2\tau_{12}\sinh 2\alpha_0 \cos 2\beta\right]\,. \end{aligned} \tag{3.6}$$

The modulus of transformation h is given by

$$h = \{\tfrac{1}{2}c^2(\cosh 2\alpha - \cos 2\beta)\}^{-\frac{1}{2}}\,,$$

and h_0 is the value of h when $\alpha = \alpha_0$.

We shall now transform (2.6) and (2.14) in elliptic co-ordinates (Love [10]):

$$U(c) = -\frac{1}{2}\int_{S_1} [\tau^{\mathrm{I}}_{\alpha\beta}(u^{\mathrm{I}}_\alpha + u^{\mathrm{II}}_\alpha) + \tau^{\mathrm{I}}_{\beta\beta}(u^{\mathrm{I}}_\beta + u^{\mathrm{II}}_\beta)] \frac{d\alpha dx_3}{h}$$
$$-\frac{1}{2}\int_{S_1} [\tau^{\mathrm{I}}_{\alpha\alpha}(u^{\mathrm{I}}_\alpha + u^{\mathrm{II}}_\alpha) + \tau^{\mathrm{I}}_{\alpha\beta}(u^{\mathrm{I}}_\beta + u^{\mathrm{II}}_\beta)] \frac{d\beta dx_3}{h}. \tag{3.7}$$

Since S_1 is the crack surface for which $\alpha=\alpha_0(=\text{constant})$, the first of these integrals is zero, and finally carrying out the integration with respect to x_3 from 0 to l, we get

$$U(c) = -\frac{l}{2}\int_{\alpha_0} [\tau^{\mathrm{I}}_{\alpha\alpha}(u^{\mathrm{I}}_\alpha + u^{\mathrm{II}}_\alpha) + \tau^{\mathrm{I}}_{\alpha\beta}(u^{\mathrm{I}}_\beta + u^{\mathrm{II}}_\beta)] \frac{d\beta}{h}. \tag{3.8}$$

Similarly (2.14) becomes

$$\frac{l}{2}\frac{\partial}{\partial c}\int_{\alpha_0} [\tau^{\mathrm{I}}_{\alpha\alpha}u^{\mathrm{II}}_\alpha + \tau^{\mathrm{I}}_{\alpha\beta}u^{\mathrm{II}}_\alpha] \frac{\alpha\beta}{h} + v\frac{\partial s_1}{\partial c} \leqslant 0. \tag{3.9}$$

The values of u^{II}_α, u^{II}_β, $\tau^{\mathrm{II}}_{\beta\beta}$ for $\alpha=\alpha_0$ are given by (stress and displacement at the hole, Poschl [11])

$$\tau^{\mathrm{II}}_{\beta\beta}(\alpha_0) = \frac{(T_1+T_2)\sinh 2\alpha_0 - (T_2-T_1)\{\cos 2\theta - e^{2\alpha_0}\cos 2(\beta-\theta)\}}{\cosh 2\alpha_0 - \cos 2\beta},$$

$$u^{\mathrm{II}}_\alpha(\alpha_0) = \frac{(1-\eta)}{2\mu h_0}[(T_1+T_2)+(T_2-T_1)e^{2\alpha_0}\cos 2\theta(1-\tfrac{1}{2}c^2h_0^2\sinh 2\alpha_0)],$$

$$u^{\mathrm{II}}_\beta(\alpha_0) = \frac{(1-\eta)}{2\mu h_0}[(T_2-T_1)e^{2\alpha_0}(\sin 2\theta + \tfrac{1}{2}c^2h_0^2\cos 2\theta\sin 2\beta)]. \tag{3.10}$$

Or, from (3.3),

$$\tau^{\mathrm{II}}_{\beta\beta}(\alpha_0) = \frac{(\tau_{11}+\tau_{22})\sinh 2\alpha_0 + (\tau_{11}-\tau_{22})(1-e^{2\alpha_0}\cos 2\beta) - 2\tau_{12}e^{2\alpha_0}\sin 2\beta}{\cosh 2\alpha_0 - \cos 2\beta},$$

$$u^{\mathrm{II}}_\alpha(\alpha_0) = \frac{(1-\eta)}{2\mu h_0}[(\tau_{11}+\tau_{22})-(\tau_{11}-\tau_{22})e^{2\alpha_0}(1-\tfrac{1}{2}c^2h_0^2\sin 2\alpha_0)],$$

$$u^{\mathrm{II}}_\beta(\alpha_0) = -\frac{(1-\eta)}{2\mu h_0}[2\tau_{12}+\tfrac{1}{2}c^2h_0^2(\tau_{11}-\tau_{22})\sin 2\beta]e^{2\alpha_0}. \tag{3.11}$$

The values of $\tau^{\mathrm{I}}_{\alpha\alpha}$, $\tau^{\mathrm{I}}_{\beta\beta}$ and u^{I}_α for $\alpha=\alpha_0$ are given by (3.6). Substituting these in (3.8) and integrating between the limits $\beta=0$ and $\beta=2\pi$, we get

$$U(c) = \frac{\pi(1-\eta)}{2\mu}c^2 l(\tau_{22}^2+\tau_{12}^2) + O\left(\frac{c^2}{\mu}l\tau_{ij}^2\sinh 2\alpha_0\right). \tag{3.12}$$

As $\alpha_0\to 0$, corresponding to a narrow crack, we have

$$U(c) = \frac{\pi(1-\eta)}{2\mu}c^2 l(\tau_{22}^2+\tau_{12}^2), \tag{3.13}$$

for plane stress and

$$U(c) = \frac{\pi c^2}{E}l(\tau_{22}^2+\tau_{12}^2), \tag{3.14}$$

for generalised plane stress. Now

$$S_1 = 4cl\cosh\alpha_0\int_0^{\pi/2}(1-\operatorname{sech}^2\alpha_0\sin^2\varphi)^{\frac{1}{2}}d\varphi, \tag{3.15}$$

Therefore, as $\alpha_0\to 0$, $S_1\to 4cl$. From (3.9) we get for plane strain

$$-\frac{l}{2}\frac{\partial}{\partial c}\left[\frac{\pi(1-\eta)}{\mu}c^2(\tau_{22}^2+\tau_{12}^2)\right] + v\frac{\partial}{\partial c}(4c)l \leqslant 0. \tag{3.16}$$

These results show that for a narrow elliptic crack, $U(c)$ or U^{II} depend only on the crack length and not on the crack shape α_0.

Results (3.13) and (3.14) agree with those of Orowan [6] (uniaxial tension normal to the crack $\tau_{12}=0$) and with Starr [12] for the case of pure shear ($\tau_{22}=0$). They also agree in the limit with the result given for a more general case by Stroh [13].

We now apply the thermodynamic approach as used by Griffith. From (3.16),

$$-\frac{\pi(1-\eta)}{\mu}c\left[\tau_{22}^2+\tau_{12}^2\right]+4\nu\leqslant 0\,, \tag{3.17}$$

and if we put $\tau_{12}=0$ (i.e. $\theta=0$) and $\tau_{22}=T_2$ (see (3.3)), we get

$$\tau_{22}=\tau_2\geqslant\left(\frac{4\nu\mu}{\pi(1-\eta)c}\right)^{\frac{1}{2}}, \tag{3.18}$$

as the condition for propagation of Griffith's crack (Orowan [6]). Writing

$$K=\left(\frac{4\nu\mu}{\pi(1-\eta)c}\right)^{\frac{1}{2}} \tag{3.19}$$

(which is known as Griffith's equation), the equation (3.17) can be written as

$$\tau_{22}^2+\tau_{12}^2\geqslant K^2\,. \tag{3.20}$$

Generally we suppose that there are cracks of all orientations with respect to the principal stress axes and we have to determine the orientation of the crack for which the term $\tau_{22}^2+\tau_{12}^2$ in (3.17) has a maximum value. From (3.3) the maximum values are given by

$$\theta=0\ \text{ or }\ \pi/2\,. \tag{3.21}$$

Thus $\tau_{22}^2+\tau_{12}^2$ has a maximum value T_1^2 when $\theta=\pi/2$, so that T_1 is normal to the crack, $T_1^2>T_2^2$; and a maximum value T_2^2 when $\theta=0$, so that T_2 is normal to the crack and $T_2^2>T_1^2$. That is, it has a maximum value when the stress of greatest magnitude is normal to the crack. However, if this stress happens to be compressive, such a crack will not propagate. Thus the minimisation of Gibb's free energy, using equation (3.20), does not lead to a satisfactory criterion for the propagation of cracks under shear.

4. Theory of maximum tensile stress close to the tip of the crack

The stress on a crack of orientation θ is $\tau_{\beta\beta}(\alpha_0)$ as given by (3.10). We shall have to solve the equation

$$\frac{\partial}{\partial\beta}\left[\tau_{\beta\beta}(\alpha_0)\right]=0\,. \tag{4.1}$$

If cracks of all orientations are present in the body and the parameter α_0 which determines the shape of the crack (i.e. the radius of curvature of the crack tip) is independent of θ, then we can also determine the orientation of the crack for which the maximum tensile stress is a maximum by the help of the equation

$$\frac{\partial}{\partial\theta}\left[\tau_{\beta\beta}(\alpha_0)\right]=0\,, \tag{4.2}$$

giving

$$\tan 2\theta=\frac{e^{2\alpha_0}\sin 2\beta}{e^{2\alpha_0}\cos 2\beta-1}\,. \tag{4.3}$$

Equation (4.1) gives

$$\sin 2\beta\left[\frac{1}{\sqrt{2}}(T_1-T_2)e^{\alpha_0}\sqrt{(\cosh 2\alpha_0-\cos 2\beta)}-(T_1+T_2)\sinh 2\alpha_0\right]=0\,. \tag{4.4}$$

One solution gives

$$\beta = 0\,,\ \theta = 0\,, \tag{4.5}$$

another gives

$$\cos 2\beta = \cosh 2\alpha_0 - 2\left(\frac{T_1+T_2}{T_1-T_2}\right)^2 e^{-2\alpha_0}\sinh^2 2\alpha_0\,, \tag{4.6}$$

and

$$\tan 2\theta = -\left\{4\left(\frac{T_1+T_2}{T_2-T_1}\right)^2 - 1\right\}^{\frac{1}{2}}\,. \tag{4.7}$$

For real solutions $\tan 2\theta$ and $\cos 2\beta$, we should have

$$4\left(\frac{T_1+T_2}{T_2-T_1}\right)^2 - 1 \geqslant 0 \ \text{ or } \ (3T_2+T_1)(T_2+3T_1) \geqslant 0\,. \tag{4.8}$$

Substituting in (3.11) the values of $\cos 2\beta$ and $\tan 2\theta$ given by (4.6) and (4.7), and taking limits for $\alpha_0 \to 0$, assuming $T_2 - T_1 > 0$ and with the convention that tensile stress is positive, we get the maximum tensile stress as follows:

(i) when $3T_2 + T_1 \geqslant 0$,

$$\left.\begin{aligned} &\tau_{\beta\beta}(\alpha_0)_{\max} = 2T_2 \,/\alpha_0 \\ &\theta = 0\,,\ \beta = 0 \end{aligned}\right\} \tag{4.9}$$

In this case T_2 is always tensile and normal to the crack and the maximum tensile stress occurs exactly at the tips of the crack.

(ii) When $3T_2 + T_1 \leqslant 0$,

$$\left.\begin{aligned} &\tau_{\beta\beta}(\alpha_0)_{\max} = -\frac{(T_2-T_1)^2}{4\alpha_0(T_1+T_2)} \\ &\cos 2\theta = -\frac{1}{2}\frac{T_2-T_1}{T_1+T_2} \\ &\sin 2\beta = 2\alpha_0\left[4\left(\frac{T_1+T_2}{T_2-T_1}\right)^2 - 1\right]^{\frac{1}{2}} \end{aligned}\right\} \tag{4.10}$$

This means that if the maximum tensile stress makes an angle φ with the major axis of the crack, then

$$\tan\varphi = -\cot 2\theta\,. \tag{4.11}$$

The crack with maximum tensile stress will propagate when this stress, given by (4.9) and (4.10), reaches a critical value T which is a physical constant for a solid.

As a particular example of case (a), let us consider the case that the body is subject to uniaxial tension. If the measured strength of the body in this case is K' (so that $T_2 = K'$), then the critical tensile stress $T = 2K'/\alpha_0$. When the maximum stress value given by (4.9) and (4.10) is equated to this value of T, we get the condition for crack propagation. This is found to be

(i) when $3T_2 + T_1 \geqslant 0$:

$$T_2 = K'\,,\ \theta = 0\,, \tag{4.12}$$

(ii) when $3T_2 + T_1 \leqslant 0$:

$$\left.\begin{aligned} &(T_2-T_1)^2 + 8K'(T_1+T_2) = 0 \\ &\cos 2\theta = -\frac{1}{2}\frac{T_2-T_1}{T_1+T_2} \end{aligned}\right\} \tag{4.13}$$

Now it is known that $\alpha_0 = (\rho/c)^{\frac{1}{2}}$, where ρ is the radius of curvature of the ellipse at the ends of the major axis (i.e. the radius of the edge of the crack), and therefore from (2.2) and (4.9)

$$2T_2(c/\rho)^{\frac{1}{2}} = T = \left(\frac{1.09\nu E}{b}\right)^{\frac{1}{2}}, \quad (\eta = 0.25),$$

or

$$T_2 = \left(\frac{1.09\nu\rho E}{bc}\right)^{\frac{1}{2}} \tag{4.14}$$

(b being the equilibrium inter-atomic distance).

It has been shown by Cottrell [7, 8] that the tips of a real crack will have a finite radius of curvature ρ and the minimum value of ρ is $0.81b$. Putting this value in (4.14) we get

$$T_2(\min) = (\nu E/4.53c)^{\frac{1}{2}}, \tag{4.15}$$

compared with Griffith's equation (3.19) obtained by minimising Gibb's free energy for the system (with $\eta = 0.25$)

$$T_2 = (\nu E/1.48c)^{\frac{1}{2}}. \tag{4.16}$$

From (4.15) we get a minimum value of the crack length c. Since $T_2(\min)$ must equal the measured tensile strength K', we get from (4.15)

$$c(\min) = \nu E/(4.53K'^2). \tag{4.17}$$

The equation (4.12) corresponds to tensile fracture normal to the major principal stress axis and equation (4.13) corresponds to shear cracks inclined to the principal stress axes. The angle θ defines the orientation of the crack that propagates and in the case of shear fractures this is not necessarily the same as the orientation of the macroscopic fracture surface since the crack does not propagate precisely at its tips (see equation (4.11)). It seems likely, as suggested by Brace and Bombalakis [14], that shear fractures are necessarily produced by the coalescence of a series of cracks, because individual cracks tend to propagate approximately normal to the least principal stress.

The first equation of (4.13) can be written as

$$\tau_{12}^2 + 4K'\tau_{22} = 4K'^2. \tag{4.18}$$

Equation (4.18) is the envelope of the stress circle in the Mohr diagram [15]. Equation (3.20) is represented by a circle of radius K with centre at origin and equation (4.28) is represented by a parabola. As long as the distance to any point on the parabola is greater than or equal to K, equation (3.20) is satisfied for that point. It can be seen that this is true for all points on the parabola if $K' \geq K$, i.e. if the measured tensile strength is greater than or equal to that given by

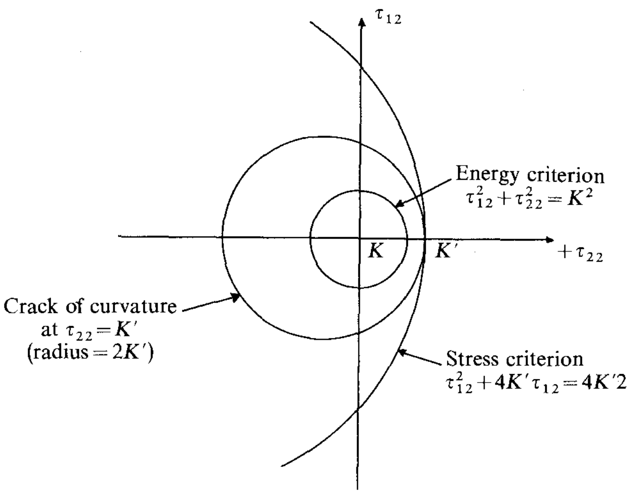

Figure 2. Mohr diagram representing two criteria for crack propagation.

Griffith's equation (3.18). The latter condition must apply, since otherwise there would not be a decrease of Gibb's free energy of the system in this case. Therefore there will always be a decrease of Gibb's free energy of the system when equations (4.12) and (4.13) are satisfied, and these equations represent the necessary and sufficient conditions for fracture to occur due to the propagation of cracks.

5. Conclusion

It is known that the strength of brittle fracture is a function of both deviator stresses and hydrostatic stress. If the stresses were tensile then tensile rupture may occur, otherwise the failure would be by shear. Mohr [15] suggested that if the strength of such a material is independent of the intermediate principal stress, then the strength is governed by a relationship between the shear stress τ_{12} and the normal stress τ_{22} on the plane of rupture or of shear given by

$$\tau_{12} = f(\tau_{22})\,.$$

Nadai [16] showed that this relationship is the envelope of the major principal stress circle in Mohr diagram and the orientation of the plane of rupture or of shear for a given stress circle is supposed to be determined by the orientation of the diameter which passes through the point of contact with the envelope.

In the case of brittle fracture, the classic works of Von Karman [17] showed that the Mohr diagram was parabolic. But the reason remained unexplained until recently. It lies in the nature of the equations (4.12), (4.13) and (4.18) governing the propagation of Griffith's crack.

Although equations (4.12) and (4.13) are derived for plane strain conditions, the general nature of the laws of brittle fracture which they predict is in good agreement for more general conditions.

Hence, using Griffith's crack we have shown that the thermodynamic criterion for crack propagation is not a sufficient one. The necessary and sufficient criterion is that there should occur at the crack edge total stresses which are sufficient to rupture the atomic bonds. Assuming that there are cracks of all orientations with respect to the applied principal stress axes, the theory leads to a parabolic relationship

$$\tau_{12}^2 + 4K'\tau_{22} = 4K'^2$$

between the shear stress τ_{12} and the normal stress τ_{22} acting on the plane containing the crack that propagates. The fact that there is more than enough energy available for the crack propagation process is perhaps the explanation of the phenomenon of explosive shattering of some of these materials when they are fractured in compression.

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